NOTES ON

Model Theory

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Based on *Model Theory: An Introduction* (GTM 217).

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Contents

Chapter 1

Structures and Theories

The definitions and preservation results for languages, structures, satisfaction, embeddings, and elementary equivalence are as in the Enderton note, §§2.1–2.2. We retain the additional examples and the study of definability and interpretations from Marker's Chapter 1. The notation appendix records the changes in conventions.

1.1 Additional examples of theories

Here a theory is any set of sentences, without a closure requirement. The basic axiomatizations of orders, graphs, groups, and fields are already in the Enderton note.

Groups and modules

In the group language $\mathcal{L}_{\mathrm{g}} = \{\cdot, e\}$, the torsion-free groups are axiomatized by the group axioms together with

$$\forall x\,(x^n = e \to x = e), \qquad n \geq 2.$$

Here x^n abbreviates a term with n factors; there is one sentence for each external integer n. Divisible groups are axiomatized by adjoining

$$\forall y \exists x\,(x^n = y), \qquad n \geq 2.$$

For a fixed N, the condition that every element has order at most N is expressed by $\forall x\ \bigvee_{n=1}^{N} x^n = e$. Allowing arbitrary finite order would require an infinite disjunction in this expression, which is not a first-order formula. This observation alone does not exclude a different axiomatization; nonaxiomatizability of the class of torsion groups requires a further argument.

An ordered Abelian group is an Abelian group in $\{+, 0\}$ with a linear order satisfying

$$\forall x \forall y \forall z\,(x < y \to x + z < y + z).$$

Such a group is torsion-free: if $a > 0$, then $na > 0$ for every $n \geq 1$, and the case $a < 0$ follows by applying this to $-a$.

Fix a ring R with identity. To describe left R-modules, use

$$\mathcal{L}_R = \{+, 0\} \cup \{\lambda_r : r \in R\},$$

where each λ_r is a unary function symbol. Besides the Abelian group axioms, require, for all $r, s \in R$,

$$\begin{aligned} \forall x \forall y \quad & \lambda_r(x + y) = \lambda_r(x) + \lambda_r(y), \\ \forall x \quad & \lambda_{r+s}(x) = \lambda_r(x) + \lambda_s(x), \\ \forall x \quad & \lambda_r(\lambda_s(x)) = \lambda_{rs}(x), \\ \forall x \quad & \lambda_1(x) = x. \end{aligned}$$

The scalars index symbols and axiom instances; the quantifiers range over module elements. In particular, the language can be uncountable.

Differential fields

In $\mathcal{L}_{\mathrm{r}} \cup \{\delta\}$, where δ is unary, add to the field axioms

$$\begin{aligned} \forall x \forall y \quad & \delta(x + y) = \delta(x) + \delta(y), \\ \forall x \forall y \quad & \delta(xy) = x\delta(y) + y\delta(x). \end{aligned}$$

Thus a differential field is a field equipped with a derivation. These axioms express additivity and the product rule, without reference to limits.

Induction as an axiom schema

In the arithmetic language $\{+, \cdot, s, 0\}$, the induction schema contains, for each formula $\varphi(v, \bar{w})$, the sentence

$$\forall \bar{w} \left[(\varphi(0, \bar{w}) \wedge \forall v \, (\varphi(v, \bar{w}) \to \varphi(s(v), \bar{w}))) \to \forall v \, \varphi(v, \bar{w}) \right].$$

Together with the usual successor and recursive addition and multiplication axioms, these are the Peano axioms. The schema says that every subset definable with parameters which contains 0 and is closed under successor is the whole domain. It does not quantify over arbitrary subsets of the domain.

1.2 Definable sets

Definition 1.1 (Marker, Definition 1.3.1). Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. A set $X \subseteq M^n$ is *A-definable*, or *definable over A*, if there are an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and a finite tuple $\bar{b}$ from A such that

$$X = \{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a}, \bar{b})\}.$$

We call X *definable* if it is M-definable. Definability without parameters is called *∅-definability*.

The empty parameter tuple is allowed. Every A-definable set is therefore definable over some finite subset of A. Equivalently, expand $\mathcal{L}$ by constants naming the elements of A and use the definition of definability from the Enderton note in this expanded structure. Every singleton $\{a\}$ is definable over $\{a\}$, although it need not be ∅-definable.

Algebraic examples

Example 1.2 (Polynomial equations and order). In a ring R, the zero set of a polynomial $p \in R[X_1, \dots, X_n]$ is definable over its coefficients. The polynomial notation abbreviates a formula in the ring language with those coefficients as parameters.
In the real field, order is already ∅-definable in the ring language:

$$x < y \quad \iff \quad \exists z\,(z \neq 0 \wedge y = x + z^2).$$

In the integer ring, Lagrange's four-square theorem gives

$$x < y \quad \iff \quad \exists z_1 \cdots \exists z_4 \left(y = x + 1 + z_1^2 + z_2^2 + z_3^2 + z_4^2\right).$$

Example 1.3 (Recovering constant fields). If F is a field, its copy inside the polynomial ring $F[t]$ is ∅-definable by

$$x = 0 \vee \exists y\,(xy = 1),$$

since the units of $F[t]$ are exactly the nonzero constants.
The constant field $\mathbb{C}$ inside $\mathbb{C}(t)$ is also ∅-definable, by

$$\exists u \exists v\,(v^2 = x \wedge u^3 + 1 = x).$$

Every constant satisfies this formula. A nonconstant solution would give a nonconstant rational map from the projective line to the elliptic curve $v^2 = u^3 + 1$, which is impossible. This example uses that algebraic-geometric fact as an input.

Proposition 1.4 (The valuation ring in $\mathbb{Q}_p$). *If p is an odd prime, the formula $\exists y\,(y^2 = 1 + px^2)$ defines $\mathbb{Z}_p$ without parameters in the field $\mathbb{Q}_p$.*

Proof. Normalize the valuation by $v_p(p) = 1$. If $v_p(a) < 0$, then

$$v_p(1 + pa^2) = 1 + 2v_p(a)$$

is odd, whereas every nonzero square has even valuation. Thus a solution forces $a \in \mathbb{Z}_p$.

Conversely, if $a \in \mathbb{Z}_p$, the polynomial $Y^2 - (1 + pa^2)$ has the simple root 1 modulo p. Hensel's lemma lifts it to a root in $\mathbb{Z}_p$. The coefficient p is a numeral, so no parameter from the field is needed. ⊣

Remark 1.5 (Arithmetic examples). Julia Robinson proved that $\mathbb{Z}$ is $\emptyset$-definable in the field $\mathbb{Q}$; Marker's example invokes this result without proving it. Another input, the Matiyasevich–Robinson–Davis–Putnam theorem, sharpens the definability of effectively enumerable relations recorded in the Enderton note: for every effectively enumerable $B \subseteq \mathbb{N}^n$, there is a polynomial $p \in \mathbb{Z}[\bar{X}, \bar{Y}]$ such that

$$\bar{a} \in B \quad \Longleftrightarrow \quad \exists \bar{b} \in \mathbb{N}^m \; p(\bar{a}, \bar{b}) = 0.$$

Negative coefficients can be moved to the other side of the equation, giving a formula in the arithmetic language. The new point is the existential polynomial form of the definition.

Closure operations

Lemma 1.6 (Marker, Lemma 1.3.3). *If $X \subseteq \mathbb{R}^n$ is A-definable in the ordered real field, then its topological closure is A-definable.*

Proof. Suppose that $\varphi(\bar{y}, \bar{a})$ defines X, with $\bar{a}$ from A. A point $\bar{x}$ belongs to its closure exactly when

$$\forall \varepsilon \left(\varepsilon > 0 \to \exists \bar{y} \left(\varphi(\bar{y}, \bar{a}) \wedge \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon \right) \right).$$ ⊣

Proposition 1.7 (Marker, Proposition 1.3.4). *The definable subsets of the finite powers of M form the smallest family containing the full powers, the interpretations of relation symbols, the graphs of function symbols, and the diagonals $\{\bar{x} : x_i = x_j\}$, and closed under:*

> (i) *finite unions, finite intersections, and complements in M^n;*
> (ii) *adjoining unused coordinates and permuting coordinates;*
> (iii) *coordinate projections;*
> (iv) *fibers $X_{\bar{b}} = \{\bar{a} : (\bar{a}, \bar{b}) \in X\}$ at arbitrary finite tuples $\bar{b}$ from M.*

Proof. All the generating sets are definable. Boolean operations correspond to connectives, adjoining or permuting coordinates to changing the list of free variables, projections to existential quantification, and fibers to substituting parameters. Thus these operations preserve definability.

For the converse, fibers of the diagonal yield all singletons, including the interpretations of constant symbols. Induction on terms produces their graphs: if $t = f(t_1, \dots, t_k)$, its graph is obtained from the graphs of $f, t_1, \dots, t_k$ by imposing

$$z_i = t_i(\bar{x})\ (1 \le i \le k), \qquad y = f(z_1, \dots, z_k),$$

and projecting away the z_i. The same construction gives the sets defined by atomic formulas. Induction on formulas then uses Boolean operations and projections; finally, taking fibers supplies the parameters. ⊣

The key correspondence is

$$\pi\big(\{(\bar{a}, b) : \mathcal{M} \models \varphi(\bar{a}, b)\}\big) = \{\bar{a} : \mathcal{M} \models \exists y\, \varphi(\bar{a}, y)\}.$$

For A-definability, include the singletons naming the language's constants among the generators and allow fibers only at tuples from A. This gives the corresponding description without introducing extra parameters.

Automorphisms and undefinability

> **Proposition 1.8** (Marker, Proposition 1.3.5)**.** *If $X \subseteq M^n$ is A-definable, every automorphism of $\mathcal{M}$ fixing A pointwise fixes X setwise.*

Proof. Write $X = \varphi(M^n, \bar{a})$ with $\bar{a}$ from A. By the isomorphism theorem in the Enderton note, for such an automorphism σ,

$$\mathcal{M} \models \varphi(\bar{b}, \bar{a}) \iff \mathcal{M} \models \varphi(\sigma(\bar{b}), \sigma(\bar{a})) \iff \mathcal{M} \models \varphi(\sigma(\bar{b}), \bar{a}).$$ ⊣

Corollary 1.9 (Marker, Corollary 1.3.6). *The set $\mathbb{R}$ is not definable in the complex field, even with parameters.*

Proof. Any proposed definition uses a finite parameter set $A \subseteq \mathbb{C}$. Choose $r \in \mathbb{R}$ and $s \in \mathbb{C} \setminus \mathbb{R}$ algebraically independent over $\mathbb{Q}(A)$. Extending transcendence bases and then passing to algebraic closures gives an automorphism of $\mathbb{C}$ fixing A and sending r to s. This contradicts proposition 1.8. $\dashv$

The converse to proposition 1.8 fails in general. Every automorphism of the real field fixes $\mathbb{Q}$ and preserves its definable order. Density of $\mathbb{Q}$ then forces it to be the identity. Hence every subset of $\mathbb{R}$ is invariant under every automorphism, but there are only $|\mathbb{R}|$ definitions with finite parameter tuples and $2^{|\mathbb{R}|}$ subsets of $\mathbb{R}$.

1.3 Interpretability

Definable interpretations

Definition 1.10. An $\mathcal{L}_0$-structure $\mathcal{N}$ is *definably interpretable* in an $\mathcal{L}$-structure $\mathcal{M}$ if there is a nonempty definable $X \subseteq M^n$, equipped with definable relations, functions, and distinguished elements for the symbols of $\mathcal{L}_0$, such that the resulting structure is isomorphic to $\mathcal{N}$.

A function is definable when its graph is definable; a distinguished element is definable when its singleton is definable. All definitions here take place in the ambient structure $\mathcal{M}$, and parameters are allowed. The languages $\mathcal{L}$ and $\mathcal{L}_0$ can differ.

For a fixed interpretation, statements about the interpreted structure translate into statements about $\mathcal{M}$: replace its basic relations and function graphs by their definitions, and restrict each quantifier to X. One variable of the interpreted structure becomes an n-tuple of ambient variables.

Example 1.11 (Matrix groups). The group $\mathrm{GL}_n(K)$ is definably interpretable in the field K. Its domain is

$$X = \{(a_{ij}) \in K^{n^2} \,:\, \det(a_{ij}) \neq 0\},$$

and the multiplication graph is defined by the polynomial equations $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. The identity matrix is definable as well. Adding finitely many polynomial equations gives the same conclusion for a linear algebraic subgroup, using its coefficients as parameters.

Example 1.12 (A structure on tuples). Let M be a countably infinite set in the empty language. On M^2, the relation

$$E((x_1, x_2), (y_1, y_2)) \iff x_1 = y_1$$

defines a structure with countably many equivalence classes, each countably infinite. Thus a structure with a nontrivial relation is definably interpretable in a pure set by using tuples.

Recovering a field from its affine group

Let F be an infinite field and

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F^\times,\ b \in F \right\}.$$

Although G is definable in F, we now work in the pure group G and recover the field using two parameters. Fix $\tau \in F \setminus \{0, 1\}$ and put

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}.$$

Their centralizers are definable over $\{\alpha, \beta\}$:

$$U = C_G(\alpha) = \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\},$$
$$D = C_G(\beta) = \left\{ d(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} : t \in F^\times \right\}.$$

Group multiplication on U gives $u(x)u(y) = u(x + y)$. For $a \in U \setminus \{I\}$, define $i(a)$ as the unique $d \in D$ such that $d^{-1}ad = \alpha$. Since

$$d(t)^{-1}u(x)d(t) = u(x/t),$$

we have $i(u(x)) = d(x)$. Define a second operation on U by

$$a * b = \begin{cases} i(b)a\,i(b)^{-1}, & b \neq I, \\ I, & b = I. \end{cases}$$

Its graph is definable in the group, and $u(x) * u(y) = u(xy)$. Consequently,

$$(F, +, \cdot, 0, 1) \cong (U, \cdot, *, I, \alpha).$$

This interpretation recovers both field operations from the group operation and the chosen parameters.

Encoding linear orders in graphs

Given a linear order $(A, <)$, form an undirected graph G_A. For each $a \in A$, attach three fresh vertices p_a, q_a, r_a with edges

$$\{a, p_a\}, \quad \{p_a, q_a\}, \quad \{p_a, r_a\}, \quad \{q_a, r_a\}.$$

For each $a < b$, attach three further fresh vertices u_{ab}, v_{ab}, w_{ab} with edges

$$\{a, u_{ab}\}, \quad \{u_{ab}, v_{ab}\}, \quad \{u_{ab}, w_{ab}\}, \quad \{v_{ab}, b\}.$$

There are no other edges. The first attachment marks the original vertices: they are precisely the vertices outside triangles which have a neighbor on a triangle. The second attachment encodes the direction from a to b; its leaf w_{ab} distinguishes its two ends.

These descriptions are first-order. For example, the order formula asks for distinct u, v, w whose neighborhoods are respectively $\{a, v, w\}$, $\{u, b\}$, and $\{u\}$. On the definable copy of A, it holds exactly when $a < b$. Thus $(A, <)$ is definably interpretable in G_A.

Marker further axiomatizes the resulting graphs by a theory T: every model is isomorphic to some G_A, and recovering the order and rebuilding its graph returns the original graph up to isomorphism (Lemma 1.3.8). The attachments can be required to be unique up to their internal symmetries. The point is that a language with one symmetric relation can encode an entire ordered structure.

Definable quotients

Definition 1.13 (Marker, Definition 1.3.9)**.** An $\mathcal{L}_0$-structure $\mathcal{N}$ is *interpretable* in $\mathcal{M}$ if there are a definable $X \subseteq M^n$ and a definable equivalence relation E on X such that X/E, equipped with structure induced by definable relations on tuples from X, is isomorphic to $\mathcal{N}$.

The defining relations must be invariant under replacing representatives by E-equivalent tuples. For a function symbol, its graph must give a unique *output class* for every tuple of input classes. Constants are represented by definable classes. A definable interpretation is the special case where E is equality. Quotients avoid requiring a definable choice of representatives.

Example 1.14 (Groups and projective space)**.** If G is a definable group and H a definable normal subgroup, then G/H is interpretable using $E(g, h)$ defined by $g^{-1}h \in H$.
In a field K, put $X = K^{n+1} \setminus \{\bar{0}\}$ and

$$\bar{a} \sim \bar{b} \quad \Longleftrightarrow \quad \exists \lambda \, (\lambda \neq 0 \wedge \bar{b} = \lambda \bar{a}).$$

Then $X/{\sim} = \mathbb{P}^n(K)$. For homogeneous f, the condition $f(\bar{a}) = 0$ descends to this quotient and defines an interpretable relation over the coefficients of f.

Example 1.15 (The value group of $\mathbb{Q}_p$). For an odd prime p, proposition 1.4 defines $\mathbb{Z}_p$ inside $\mathbb{Q}_p$. Its unit group

$$V = \{x \in \mathbb{Z}_p \,:\, \exists y \in \mathbb{Z}_p\ (xy = 1)\}$$

is definable. The valuation induces an isomorphism

$$\mathbb{Q}_p^\times / V \longrightarrow (\mathbb{Z}, +), \qquad xV \longmapsto v_p(x).$$

Moreover, $v_p(x) \geq v_p(y)$ exactly when $x/y \in \mathbb{Z}_p$. Hence the ordered additive group of integers is interpretable in the field. Division here abbreviates its definable graph in the ring language.

1.4 Imaginary sorts

The many-sorted formalism is already available in the Enderton note, §4.3. Its new use here is to make definable equivalence classes into elements of additional sorts.

For each $\emptyset$-definable equivalence relation E on some M^n, add a sort S_E with universe M^n/E and a quotient map

$$\pi_E \,:\, M^n \longrightarrow S_E, \qquad \bar{a} \longmapsto [\bar{a}]_E.$$

Retain the original structure on the equality sort $S_=$, identified with M. The resulting many-sorted structure is denoted $\mathcal{M}^{\mathrm{eq}}$; elements of its quotient sorts are called *imaginaries*. The same parameter-free formulas specify the corresponding sorts in elementarily equivalent structures.

Lemma 1.16 (Marker, Lemma 1.3.10).

(i) If $X \subseteq M^n$ is definable in $\mathcal{M}^{\mathrm{eq}}$, allowing parameters from any sort, then it is definable in $\mathcal{M}$.

(ii) If $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{M}^{\mathrm{eq}} \equiv \mathcal{N}^{\mathrm{eq}}$.

(iii) Every structure interpretable in $\mathcal{M}$ is definably interpretable in $\mathcal{M}^{\mathrm{eq}}$.

(iv) Every automorphism of $\mathcal{M}^{\mathrm{eq}}$ restricts to an automorphism of $\mathcal{M}$, and every automorphism σ of $\mathcal{M}$ extends uniquely by

$$[\bar{a}]_E \longmapsto [\sigma(\bar{a})]_E.$$

Replacing imaginary variables and parameters by real tuples of representatives explains parts (i) and (ii); part (i) may use new real parameters. For part (iii), parameter-dependent quotients are accommodated by including their parameters as extra coordinates before taking a quotient sort.

Appendix A

Notation and conventions

Comparisons below refer to the accompanying Enderton *note*. Objects already defined there keep their meaning unless a change is stated explicitly. Numbering of statements in this note is local; headings such as "Marker, Proposition 1.3.4" identify the corresponding result in the source text.

A.1 Structures, tuples, and satisfaction

$\mathcal{L}$

A language specifies its nonlogical function, relation, and constant symbols, with their arities. Equality is logical and is always available here. Thus $\mathcal{L} = \emptyset$ is the language of pure sets with equality. Unlike the bookkeeping in the Enderton note, $\forall$ is not listed among the nonlogical symbols.

$\mathcal{M}$ and M

The calligraphic letter denotes a structure; the ordinary letter denotes its universe. Thus M corresponds to $|\mathfrak{A}|$ in the Enderton note when $\mathcal{M}$ corresponds to $\mathfrak{A}$. Here $|M|$ is a cardinality, not a universe. The notations $f^{\mathcal{M}}$, $R^{\mathcal{M}}$, and $c^{\mathcal{M}}$ denote symbol interpretations as before.

$\bar{a}$, $\bar{x}$, and $A^{<\omega}$

A bar denotes a finite tuple; its length is determined by context. Marker writes $\bar{a} \in A$ for a finite tuple from A, and sets $A^{<\omega} = \bigcup_{n \geq 1} A^n$. We write "$\bar{a}$ from A" when the length is unimportant. An empty tuple is also allowed when a definition uses no parameters. A map acts on tuples coordinatewise: $\sigma(\bar{a}) = (\sigma(a_1), \dots, \sigma(a_n))$.

$t^{\mathcal{M}}(\bar{a})$

The value of a term at a tuple. This is the value denoted $\bar{s}(t)$ in the Enderton note when the assignment s sends the displayed variables to $\bar{a}$.

$\mathcal{M} \models \varphi(\bar{a})$

Satisfaction with the displayed free variables assigned $\bar{a}$. This corresponds to $\models_{\mathfrak{A}} \varphi[\![\bar{a}]\!]$ in the Enderton note. Entries of $\bar{a}$ need not be named by terms of $\mathcal{L}$; the notation specifies an assignment. For a sentence σ, write simply $\mathcal{M} \models \sigma$.

$\varphi(M^n, \bar{b})$
The set $\{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a}, \bar{b})\}$. We also abbreviate blocks of quantifiers by $\exists\bar{x}$ and $\forall\bar{x}$.

A.2 Maps and theories

Embeddings and isomorphisms. An $\mathcal{L}$-embedding is the injective map called an *isomorphic embedding*, or an *isomorphism into*, in the Enderton note. It preserves functions and constants and both preserves and reflects relations. Marker reserves *isomorphism* for a bijective embedding. In particular, $\mathcal{M} \cong \mathcal{N}$ has the same meaning in both notes. Inclusion as a substructure also has the same meaning.

Theories. Here an $\mathcal{L}$-theory T is any set of $\mathcal{L}$-sentences, possibly unsatisfiable. In the Enderton note, a theory must satisfy $T = \mathrm{Cn}(T)$. To translate the present convention into that one, replace T by

$$\mathrm{Cn}(T) = \{\sigma : T \models \sigma\}.$$

The model class and logical consequences do not change. The notations $\mathrm{Mod}(T)$, $T \models \sigma$, and

$$\mathrm{Th}(\mathcal{M}) = \{\sigma : \mathcal{M} \models \sigma\}$$

retain their meaning; $\mathrm{Th}(\mathcal{M})$ is closed under consequence under either convention. Likewise $\mathcal{M} \equiv \mathcal{N}$ still means $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Elementary classes and axiomatizations. Marker calls $\mathrm{Mod}(T)$ an *elementary class* for any set T of sentences. This is EC_Δ in the Enderton note. Enderton's narrower EC means axiomatizability by one sentence, equivalently by finitely many sentences. In this chapter, an *axiomatization* need not be effective; the Enderton note uses *axiomatizable theory* for a consequence-closed theory generated by a decidable set of axioms.

A.3 Definability and parameters

The word *parameter* here means an element of a structure used in a formula, as in $\varphi(\bar{x}, \bar{b})$. The Enderton note also uses this word for nonlogical symbols; those are called *symbols* here. Naming an element by a newly added constant connects the two usages, but the element and its name remain distinct.

Definable
Definable using a finite tuple of parameters from the ambient structure. This is broader than the default definition in the Enderton note.

A-**definable**
Definable using a finite tuple from $A \subseteq M$.

$\emptyset$-**definable**
Definable without parameters from M. This is exactly the Enderton note's default notion. Constants already belonging to $\mathcal{L}$ remain available.

$\mathrm{Aut}(\mathcal{M}/A)$
The group of automorphisms of $\mathcal{M}$ fixing each element of A. Such automorphisms preserve an A-definable set *setwise*; they need not fix every element of that set.

A.4 Recurring languages and quotients

The ring language is $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$, with binary subtraction as in Marker. The ordered ring language is $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_{\mathrm{r}} \cup \{<\}$. The group language is $\mathcal{L}_{\mathrm{g}} = \{\cdot, e\}$; the identity is also written 1 or I in examples. Powers and polynomial expressions abbreviate finite terms, with coefficients supplied as parameters when necessary.

The Enderton note often uses $\{0, 1, +, \cdot\}$ for fields. Adding subtraction does not change definable sets, since $x - y = z$ is equivalent to $x = y + z$. For modules, we write λ_r for the scalar-multiplication symbol that Marker denotes by r.

The notation ACF denotes the field axioms together with the axiom, for each $n \geq 1$, that every monic polynomial of degree n has a root. For a prime p, ACF_p adds $p \cdot 1 = 0$; ACF_0 adds $p \cdot 1 \neq 0$ for every prime p. The characteristic-zero case is already treated in the Enderton note; here these symbols denote axiom sets rather than their consequence closures.

For an equivalence relation E on X, write $[a]_E$ (also a/E) for an equivalence class and X/E for the quotient. In an interpretation, X can itself be a subset of a finite power M^n. A *definable interpretation* uses X as its domain; a general *interpretation* may use X/E.

The superscript in $\mathcal{M}^{\mathrm{eq}}$ denotes the many-sorted expansion by imaginary sorts, not a power or an elementary-equivalence class. For a parameter-free definable equivalence relation E on M^n, the sort S_E has universe M^n/E, with quotient map π_E. Its equality sort recovers M.